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Squeezed states of the generalized minimum uncertainty state for the Caldirola–Kanai Hamiltonian

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Abstract

We show that the ground state of the well-known pseudo-stationary states for the Caldirola–Kanai Hamiltonian is a generalized minimum uncertainty state, which has the minimum allowed uncertainty $\Delta q \Delta p = \hbar\sigma_0/2$, where σ_0 ($\geqslant 1$) is a constant depending on the damping factor and natural frequency. The most general symmetric Gaussian states are obtained as the one-parameter squeezed states of the pseudo-stationary ground state. It is further shown that the coherent states of the pseudo-stationary ground state constitute another class of the generalized minimum uncertainty states.

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1. Introduction

The Hamiltonian for a harmonic oscillator with an exponentially increasing mass has been introduced by Caldirola and Kanai [1] and the corresponding Lagrangian by Bateman [2]. The fact that its classical motion describes a damping motion has motivated the investigation of the Caldirola–Kanai (CK) Hamiltonian as a quantum damped system [3]. The pseudo-stationary states of the CK Hamiltonian have been found in many different ways [4–17]. In particular, the invariant operator method provides a convenient tool to find exact wavefunctions for such time-dependent oscillators [18]. However, there have been debates whether this quantum oscillator genuinely describes a dissipative system or not [4, 6, 7, 19].

In this paper we show that all the Gaussian states of the CK Hamiltonian with $\langle \hat{q} \rangle = 0 = \langle \hat{p} \rangle$ satisfy the generalized minimum uncertainty relation

$$\Delta q \Delta p \geqslant \frac{\hbar}{2}\sigma_0 \qquad (\sigma_0 \geqslant 1) \tag{1}$$

where $\sigma_0 = 1\big/\left(1 - \gamma^2\big/4\omega_0^2\right)^{1/2}$ is a constant depending on the damping factor γ and the natural frequency ω_0. It is shown that the pseudo-stationary ground state is the generalized minimum uncertainty state (GMUS), a generalization of the minimum uncertainty state with $\sigma_0 = 1$ [20]. Using the linear invariant operators [21–23], we find the most general Gaussian states for the CK Hamiltonian, which have zero moment of position and momentum, and show that the pseudo-stationary ground state is, in fact, a GMUS. The GMUS that is symmetric about the origin is interpreted as the vacuum state of a time-dependent oscillator in [22]. We further show that the coherent states of the pseudo-stationary ground state are also the GMUSs.

2. Squeezed states of pseudo-stationary states

The harmonic oscillator with an exponentially increasing mass $m = m_0 \, \mathrm{e}^{\gamma t}$ has the CK Hamiltonian

$$\hat{H}(t) = \frac{1}{2m_0} \, \mathrm{e}^{-\gamma t} \hat{p}^2 + \frac{m_0 \omega_0^2}{2} \, \mathrm{e}^{\gamma t} \hat{q}^2. \tag{2}$$

The Hamilton equations describe a classical damped motion

$$\ddot{u} + \gamma \dot{u} + \omega_0^2 u = 0. \tag{3}$$

Now we use the invariant operator method to find exact quantum states of the time-dependent CK Hamiltonian. For each complex solution u of equation (3), one can introduce a pair of linear invariant operators [23],

$$\begin{aligned} \hat{a}(t) &= \frac{\mathrm{i}}{\sqrt{\hbar}} [u^*(t)\hat{p} - m_0 \, \mathrm{e}^{\gamma t} \dot{u}^*(t) \hat{q}] \\ \hat{a}^\dagger(t) &= -\frac{\mathrm{i}}{\sqrt{\hbar}} [u(t)\hat{p} - m_0 \, \mathrm{e}^{\gamma t} \dot{u}(t) \hat{q}]. \end{aligned} \tag{4}$$

In fact, these operators can be made the time-dependent annihilation and creation operators satisfying the standard commutation relation at equal time:

$$[\hat{a}(t), \hat{a}^\dagger(t)] = 1 \tag{5}$$

by imposing the Wronskian condition

$$m_0 \, \mathrm{e}^{\gamma t} [u(t)\dot{u}^*(t) - u^*(t)\dot{u}(t)] = \mathrm{i}. \tag{6}$$

We note that the eigenfunctions of $\hat{a}^\dagger(t)\hat{a}(t)$, another invariant operator [23],

$$\Psi_n(q,t) = \left(\frac{1}{2^n n! \sqrt{2\pi\hbar u^* u}} \right)^{1/2} \left(\frac{u}{\sqrt{u^* u}} \right)^{n+1/2} H_n \left(\frac{q}{\sqrt{2\hbar u^* u}} \right) \exp\left[\frac{\mathrm{i} m_0 \, \mathrm{e}^{\gamma t} \dot{u}^*}{2\hbar u^*} q^2 \right] \tag{7}$$

are the exact quantum state of the Schrödinger equation [18]. Hence the task to find the general wavefunctions is equivalent to finding the general solutions to equation (3). Our stratagem is to select a complex solution u_0 satisfying equation (6) and, as equation (3) is linear, to find the general solution as a linear superposition of u_0 and u_0^*.

For underdamped motion ($\omega_0 > \gamma/2$), we select the solution

$$u_0(t) = \frac{\mathrm{e}^{-\gamma t/2}}{\sqrt{2 m_0 \omega}} \, \mathrm{e}^{-\mathrm{i}\omega t} \qquad \omega = \sqrt{\omega_0^2 - \frac{\gamma^2}{4}}. \tag{8}$$

Then the wavefunctions of number states with the solution (8) substituted into equation (7) yield the pseudo-stationary states [4–17]

$$\begin{aligned} \Psi_n(q,t) = \frac{1}{\sqrt{2^n n!}} \left(\frac{m_0 \omega \, \mathrm{e}^{\gamma t}}{\pi\hbar} \right)^{1/4} \mathrm{e}^{-\mathrm{i}\omega t (n+1/2)} H_n \left(\sqrt{\frac{m_0 \omega}{\hbar}} \, \mathrm{e}^{\gamma t} q \right) \\ \times \exp\left[-\frac{m_0 \omega \, \mathrm{e}^{\gamma t}}{2\hbar} \left(1 + \mathrm{i} \frac{\gamma}{2\omega} \right) q^2 \right]. \end{aligned} \tag{9}$$

Now, the general complex solutions satisfying the quantization condition (6) are written as

$$u_r(t) = \mu u_0(t) + \nu u_0^*(t) \tag{10}$$

where

$$|\mu|^2 - |\nu|^2 = 1. \tag{11}$$

The complex μ and ν have four real parameters, one of which is constrained by equation (11), and another of which can be absorbed into the overall phase of u_r and hence does not change the wavefunctions. However, the relative phase between μ and ν is not determined by constraints. The squeezing parameters r and ϕ in the form

$$\mu = \cosh r \qquad \nu = \mathrm{e}^{\mathrm{i}\phi} \sinh r \tag{12}$$

are, in fact, two integration constants of the second-order equation (3). Conversely, given any complex solution u satisfying equation (6), we can find the corresponding parameters μ and ν or r and ϕ. Therefore, the most general solution to equation (3) can be written as

$$u_{r\phi}(t) = (\cosh r) u_0(t) + (\mathrm{e}^{\mathrm{i}\phi} \sinh r) u_0^*(t). \tag{13}$$

That r and ϕ are the squeezed parameters is understood from the Bogoliubov transformation

$$\begin{aligned} \hat{a}_{r\phi}(t) &= \mu^* \hat{a}_0(t) - \nu^* \hat{a}_0^\dagger(t) \\ \hat{a}_{r\phi}^\dagger(t) &= \mu \hat{a}_0^\dagger(t) - \nu \hat{a}_0(t) \end{aligned} \tag{14}$$

which is obtained by substituting equation (13) into equation (4). The Bogoliubov transformation is a unitary transformation of $\hat{a}_0(t)$ and $\hat{a}_0^\dagger(t)$:

$$\begin{aligned} \hat{a}_{r\phi}(t) &= \hat{U}(z,t) \hat{a}_0(t) \hat{U}^\dagger(z,t) \\ \hat{a}_{r\phi}^\dagger(t) &= \hat{U}(z,t) \hat{a}_0^\dagger(t) \hat{U}^\dagger(z,t) \end{aligned} \tag{15}$$

where

$$\hat{U}(t,z) = \exp\left[\tfrac{1}{2}\left(z\hat{a}_0^{\dagger 2}(t) - z^* \hat{a}_0^2(t)\right)\right] \qquad z = \mathrm{e}^{\mathrm{i}(\phi+\pi)} r \tag{16}$$

is the squeeze operator [20].

Each pair of squeeze parameters r and ϕ defines a family of the invariant number operators

$$\hat{N}_{r\phi}(t) = \hat{a}_{r\phi}^\dagger(t) \hat{a}_{r\phi}(t). \tag{17}$$

The number states

$$\hat{N}_{r\phi}(t)|n, r, \phi, t\rangle = n|n, r, \phi, t\rangle \tag{18}$$

lead to the exact wavefunctions (7) for the Schrödinger equation in the form

$$\Psi_n(q,t,r,\phi) = \frac{1}{\sqrt{2^n n!}} \left(\frac{A_{r\phi}}{\sqrt{\pi}}\right)^{1/2} \exp(-\mathrm{i}\Theta_{r\phi}(n+1/2)) H_n(A_{r\phi} q) \exp(-B_{r\phi} q^2) \tag{19}$$

where

$$\begin{aligned} A_{r\phi} &= \frac{1}{\sqrt{2\hbar u_{r\phi}^* u_{r\phi}}} = \sqrt{\frac{m_0 \omega\, \mathrm{e}^{\gamma t}}{\hbar}} \frac{1}{[\cosh 2r + \sinh 2r \cos(2\omega t + \phi)]^{1/2}} \\ B_{r\phi} &= \frac{\mathrm{i} m \dot{u}_{r\phi}^*}{2\hbar u_{r\phi}^*} = \frac{m_0 \omega\, \mathrm{e}^{\gamma t}}{2\hbar} \left[\frac{\cosh r\, \mathrm{e}^{\mathrm{i}\omega t} - \mathrm{e}^{-\mathrm{i}\phi} \sinh r\, \mathrm{e}^{-\mathrm{i}\omega t}}{\cosh r\, \mathrm{e}^{\mathrm{i}\omega t} + \mathrm{e}^{-\mathrm{i}\phi} \sinh r\, \mathrm{e}^{-\mathrm{i}\omega t}} + \mathrm{i}\frac{\gamma}{2\omega}\right] \\ \Theta_{r\phi} &= \frac{\sin \omega t - \tanh r \sin(\omega t + \phi)}{\cos \omega t + \tanh r \cos(\omega t + \phi)}. \end{aligned} \tag{20}$$

Here $\Theta_{r\phi}$ is the negative phase of $u_{r\phi}$, that is, $u_{r\phi} = \rho_r \exp(-i\Theta_{r\phi})$. The wavefunctions (19), which are symmetric about the origin ($\langle\hat{q}\rangle = \langle\hat{p}\rangle = 0$), are the squeezed states of the pseudo-stationary states (9). Besides the zero squeezing parameter ($r = 0$) leading to the pseudo-stationary states, other interesting squeezing parameters

$$\cosh 2r_0 = 1 + \frac{\gamma^2}{8\omega^2} \qquad \tan\phi_0 = \frac{4\omega}{\gamma} \tag{21}$$

lead to simple harmonic wavefunctions at $t = 0$:

$$\Psi_n(q, t = 0, r_0, \phi_0) = \exp\left[-i\frac{\gamma}{4\omega}\left(n + \frac{1}{2}\right)\right] \times \left\{\frac{1}{\sqrt{2^n n!}}\left(\frac{m_0\omega}{\pi\hbar}\right)^{1/4} H_n\left(\sqrt{\frac{m_0\omega}{\hbar}}q\right)\exp\left[-\frac{m_0\omega}{2\hbar}q^2\right]\right\}. \tag{22}$$

The wavefunctions (19), evolving the harmonic wavefunctions of an undamped ($\gamma = 0$) oscillator at $t = 0$, differ from those in [11] only by the constant phase factor in equation (22).

3. Generalized minimum uncertainty state

We now find the GMUS satisfying the equality in equation (1) among the wavefunctions (19), which are symmetric about the origin. The wavefunctions (19) have the uncertainty

$$\begin{aligned}(\Delta q)_{nr\phi}(\Delta p)_{nr\phi} &= \langle n, r, \phi, t|\hat{q}^2|n, r, \phi, t\rangle^{1/2}\langle n, r, \phi, t|\hat{p}^2|n, r, \phi, t\rangle^{1/2} \\ &= \frac{\hbar}{2}\sec\left(\frac{\vartheta_\gamma}{2}\right)[\{\cosh 2r + \sinh 2r\cos(2\omega t + \phi)\} \\ &\quad \times \{\cosh 2r - \sinh 2r\cos(2\omega t + \phi + \vartheta_\gamma)\}]^{1/2}\left(n + \tfrac{1}{2}\right)\end{aligned} \tag{23}$$

where

$$\vartheta_\gamma = \sin^{-1}\left(\frac{\gamma/\omega}{1 + \gamma^2/4\omega^2}\right) = \cos^{-1}\left(\frac{1 - \gamma^2/4\omega^2}{1 + \gamma^2/4\omega^2}\right) \qquad (\pi > \vartheta_\gamma \geqslant 0). \tag{24}$$

Using equation (23) we find the condition leading to the minimum allowed uncertainty. First, from $(\Delta q)_{nr\phi}(\Delta p)_{nr\phi} = (\Delta q)_{0r\phi}(\Delta p)_{0r\phi}(n + 1/2)$, the ground state ($n = 0$) has the lower uncertainty than other excited states ($n \geqslant 1$). Second, for the zero squeezing parameter ($r = 0$), the pseudo-stationary ground state $\Psi_0(q, t)$ has the generalized minimum uncertainty at all times

$$(\Delta q)_{00\phi}(\Delta p)_{00\phi} = \frac{\hbar}{2}\sec\left(\frac{\vartheta_\gamma}{2}\right). \tag{25}$$

Thus the generalized minimum uncertainty (1) is satisfied for

$$\sigma_0 = \sec\left(\frac{\vartheta_\gamma}{2}\right) = \frac{1}{\left(1 - \gamma^2/4\omega_0^2\right)^{1/2}}. \tag{26}$$

Note that the generalized minimum uncertainty approaches the usual minimum uncertainty ($\hbar/2$) in the weak damping limit ($\gamma/\omega_0 \ll 1$). Similarly the time averaged uncertainty is

$$\begin{aligned}\overline{(\Delta q)_{0r\phi}(\Delta p)_{0r\phi}} &= \frac{\hbar}{2}\sec\left(\frac{\vartheta_\gamma}{2}\right)\left(\cosh^2 r - \frac{\cos\vartheta_\gamma}{2}\sinh^2 r\right) \\ &\geqslant \frac{\hbar}{2}\sec\left(\frac{\vartheta_\gamma}{2}\right)\end{aligned} \tag{27}$$

where the equality holds for $r = 0$. Third, in the case of zero damping ($\gamma = 0 = \vartheta_\gamma$), the CK Hamiltonian (2) is just a simple (time-independent) harmonic oscillator. Then the uncertainty relation of $\hat{q}$ and $\hat{p}$ in the state (19) is given by

$$\begin{aligned}(\Delta q)_{0r\phi}(\Delta p)_{0r\phi} &= \frac{\hbar}{2}[\cosh^2(2r) - \sinh^2(2r)\cos^2(2\omega t + \phi)]^{1/2} \\ &\geqslant \frac{\hbar}{2}.\end{aligned} \tag{28}$$

The generalized minimum uncertainty is achieved either for zero squeezing ($r = 0$) at all times or when $\cos(2\omega t + \phi) = \pm 1$. Therefore, we conclude that the pseudo-stationary ground state, which is provided by the zero squeezing ($r = 0$) solution u_0 in equation (8), gives rise to the GMUS with the centre at the origin. In particular, this GMUS is interpreted as the vacuum state in [22]. Finally we obtain the Hamiltonian expectation value

$$\langle \hat{H} \rangle_{nr\phi} = \frac{\hbar\omega}{2}\sec^2\left(\frac{\vartheta_\gamma}{2}\right)\left[\cosh 2r + \sinh 2r \sin\left(\frac{\vartheta_\gamma}{2}\right)\sin\left(2\omega t + \phi + \frac{\vartheta_\gamma}{2}\right)\right]\left(n + \frac{1}{2}\right). \tag{29}$$

The time averaged $\overline{\langle \hat{H} \rangle_{nr\phi}}$ has the minimum value for $n = r = 0$, coinciding with the generalized minimum uncertainty.

There is another class of GMUSs. It is known that, for a time-independent oscillator, the coherent states of the vacuum state also have the minimum uncertainty [20]. Now, for the CK Hamiltonian, we either follow the definition of coherent states [24, 25]

$$\hat{a}_{r\phi}(t)|\alpha, r, \phi, t\rangle = \alpha|\alpha, r, \phi, t\rangle \tag{30}$$

for any complex α or apply the displacement operator to the ground state in equation (19)

$$|\alpha, r, \phi, t\rangle = \exp\big(\alpha\hat{a}^\dagger_{r\phi}(t) - \alpha^*\hat{a}_{r\phi}(t)\big)|0, r, \phi, t\rangle. \tag{31}$$

Then the generalized coherent states have the expectation values

$$\begin{aligned} q_c(t) &= \langle \alpha, r, \phi, t|\hat{q}|\alpha, r, \phi, t\rangle = \sqrt{\hbar}(\alpha u_{r\phi} + \alpha^* u^*_{r\phi}) \\ p_c(t) &= \langle \alpha, r, \phi, t|\hat{p}|\alpha, r, \phi, t\rangle = \sqrt{\hbar} m_0\, \mathrm{e}^{\gamma t}(\alpha\dot{u}_{r\phi} + \alpha^*\dot{u}^*_{r\phi}). \end{aligned} \tag{32}$$

Here q_c and p_c describe a trajectory in the phase space for each choice of α and $u_{r\phi}$. Replacing the complex α by two real variables q_c and p_c, we obtain the wavefunctions for the coherent states

$$\Psi(q, t, r, \phi, q_c, p_c) = \left(\frac{A_{r\phi}}{\sqrt{\pi}}\right)^{1/2} F_{r\phi}\exp(-\mathrm{i}\Theta_{r\phi}/2)\exp(-B_{r\phi}(q - q_c)^2)\exp(\mathrm{i}p_c q/\hbar) \tag{33}$$

where $A_{r\phi}$, $B_{r\phi}$ and $\Theta_{r\phi}$ are given in equation (20) and $F_{r\phi}$ is the additional phase factor

$$F_{r\phi} = \exp\left[\frac{\mathrm{i}}{2\hbar\dot{u}^*_{r\phi}u^*_{r\phi}}\big(u^{*2}_{r\phi}p_c^2 - 2\dot{u}^*_{r\phi}u^*_{r\phi}p_c q_c\big)\right]. \tag{34}$$

It then follows that any coherent state (33) has the same uncertainty as the general Gaussian state with $n = 0$ in equation (19):

$$\begin{aligned}(\Delta q)_{\alpha r\phi}(\Delta p)_{\alpha r\phi} &= \langle \alpha, r, \phi, t|(\hat{q} - \langle\hat{q}\rangle)^2|\alpha, r, \phi, t\rangle^{1/2}\langle \alpha, r, \phi, t|(\hat{p} - \langle\hat{p}\rangle)^2|\alpha, r, \phi, t\rangle^{1/2} \\ &= (\Delta q)_{0r\phi}(\Delta p)_{0r\phi}.\end{aligned} \tag{35}$$

This implies that the coherent states of the GMUS are also GMUSs. Thus the coherent states of the pseudo-stationary ground state constitute a family of GMUSs, which is the time-dependent generalization of a time-independent oscillator [20].

4. Conclusion

We have shown that the Caldirola–Kanai Hamiltonian satisfies the generalized minimum uncertainty $\Delta q \Delta p \geqslant \hbar\sigma_0/2$ for $\sigma_0 = 1\big/\big(1-\gamma^2\big/4\omega_0^2\big)^{1/2}$, where γ is the damping factor and ω_0 is the natural frequency. It is found that the well-known pseudo-stationary ground state has in fact the generalized minimum uncertainty. As the generalized minimum uncertainty state is uniquely selected for the Caldirola–Kanai Hamiltonian, this pseudo-stationary ground state may be interpreted as the vacuum state [22]. A one-parameter family of squeezed states of the pseudo-stationary states is obtained as the most general states with zero moment of position and moment. Further, it is shown that the coherent states of the pseudo-stationary ground state are the generalized minimum uncertainty states.

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